

## Another Proof of Darboux's Theorem

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**ABSTRACT.** We know that a continuous function on a closed interval satisfies the Intermediate Value Property. Likewise, the derivative function of a differentiable function on a closed interval satisfies the IVP property which is known as the Darboux's theorem in any real analysis course. Most of the proofs found in the literature use the Extreme Value Property of a continuous function. In this paper, I am going to present a simple and elegant proof of the Darboux's theorem using the Intermediate Value Theorem and the Rolles theorem

### 1. Some Preliminary Background and Known Proofs

In this section we state the Darboux's theorem and give the known proofs from various literatures.

A function  $f: [a, b] \rightarrow \mathbb{R}$  is said to satisfy the intermediate value property on  $[a, b]$  if for every  $\lambda$  between  $f(a)$  and  $f(b)$ , there exists  $c \in (a, b)$  such that  $f(c) = \lambda$ . We know, from intermediate value theorem, that a function  $f: [a, b] \rightarrow \mathbb{R}$  continuous on  $[a, b]$  satisfies the intermediate value property on  $[a, b]$ . That is

continuous function  $\Rightarrow$  Intermediate Value Property.

However, the converse of the intermediate value theorem is not necessarily true. The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(t) = \begin{cases} \sin\left(\frac{1}{t}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases}$$

which is discontinuous at  $t = 0$ , provides a counter example. Indeed, the French mathematician Darboux, proves in 1875 that derivative function of a differentiable function satisfies the intermediate value property [5]. Darboux also provides an example of a differentiable function with discontinuous derivative.

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2010 *Mathematics Subject Classification.* Primary 54C30 Real-valued Function; Secondary 26A06.

*Key words and phrases.* monotonicity, continuity, differentiability, Intermediate value property .

The statement of the Darboux's theorem follows here.

**THEOREM 1.1** (Darboux's Theorem). *If  $f$  is differentiable on  $[a, b]$  and if  $\lambda$  is a number between  $f'(a)$  and  $f'(b)$ , then there is at least one point  $c \in (a, b)$  such that  $f'(c) = \lambda$ .*

This indicates clearly that the converse of the intermediate value theorem is not necessarily true because derivative function of a differentiable function is not necessarily continuous. For example

$$f(t) = \begin{cases} t^2 \sin\left(\frac{1}{t}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0 \end{cases}$$

is continuous and has discontinuous derivative which satisfies the intermediate value property by the Darboux's theorem 1.1. Thus Darboux's provides a huge class of functions satisfying the intermediate value property, namely the derivative functions of differentiable functions. The main purpose of this short note is to provide an alternate proof of this theorem different from the standard proofs found in many textbooks on real analysis.

The standard proofs found in various literatures goes as follows:

**PROOF.** Suppose that  $f'(a) < \lambda < f'(b)$ . Let  $F : [a, b] \rightarrow \mathbb{R}$  be defined by  $F(x) = f(x) - \lambda x$  so that  $F'(x) = f'(x) - \lambda$ . Then  $F$  is differentiable on  $[a, b]$  because so is the function  $f$  by hypothesis. We find  $F'(a) = f'(a) - \lambda < 0$  and  $F'(b) = f'(b) - \lambda > 0$ . Note that  $F'(a) < 0$  means  $F(t_1) < F(a)$  for some  $t_1 \in (a, b)$ . Also, for  $F'(b) > 0$ , we can find  $t_2 \in (a, b)$  such that  $F(t_2) < F(b)$ . Thus neither  $a$  nor  $b$  can be a point where  $F$  attains absolute minimum. Since  $F$  is continuous on  $[a, b]$ , it must attain its relative minimum at some point  $c \in (a, b)$  by Extreme value theorem for a continuous function. This means that  $F'(c) = 0$  by Fermat's theorem and therefore  $f'(c) = \lambda$  as desired. The proof follows by similar arguments if  $f'(b) < \lambda < f'(a)$ .  $\square$

The above proof can be found in various textbooks of undergraduate level real analysis course including W. Rudin [11], M. Spivak [13], Bartle and Sherbert [2], K. Ross [10], W. R. Wade [14], J. M. Howe [7], T. M. Apostol [1], R. Boas [3], G. Darboux [5], S. Krantz [8]. It is left as an exercise with some hints in some of the textbooks while others give incomplete proofs leaving readers to fill in the gaps. The main trouble is in the argument that the function  $F$  can attain minimum neither at  $x = a$  nor at  $x = b$ . Some authors provide the  $\epsilon - \delta$  proof for this part of the argument. Lars Oslen, in his paper [9], makes the following comment:

"Most students typically either think that this is obvious (and that the lecture is being over pedantic by insisting on a proof), or they see the need for a proof but find the  $\epsilon - \delta$ -gymnastics in the proof less than convincing."

Lars Oslen then provides an new proof of the Darboux's theorem based only on the Mean Value theorem for a differentiable function and intermediate value theorem for a continuous function. The author, Lars Oslen, claims that his proof

is more convincing than the standard proofs found in many textbooks. I am going to present his proof as it is from his paper [9].

**Proof of Darboux's Theorem by Lars Oslen [9]:**

We may clearly assume that  $y$  lies strictly between  $f'(a)$  and  $f'(b)$ . Define continuous functions  $f_a, f_b: [a, b] \rightarrow \mathbb{R}$  by

$$f_a(t) = \begin{cases} f'(a) & \text{for } t = a, \\ \frac{f(a) - f(t)}{a - t} & \text{for } t \neq a, \end{cases}$$

and

$$f_b(t) = \begin{cases} f'(b) & \text{for } t = b, \\ \frac{f(t) - f(b)}{t - b} & \text{for } t \neq b. \end{cases}$$

It follows that  $f_a(a) = f'(a)$ ,  $f_a(b) = f_b(a)$ , and  $f_b(b) = f'(b)$ . Hence,  $y$  lies between  $f_a(a)$  and  $f_a(b)$ , or  $y$  lies between  $f_b(a)$  and  $f_b(b)$ .

If  $y$  lies between  $f_a(a)$  and  $f_a(b)$ , then (by continuity of  $f_a$ ) there exists  $s$  in  $(a, b]$  with

$$y = f_a(s) = \frac{f(s) - f(a)}{s - a}.$$

The mean value theorem ensures that there exists  $x$  in  $[a, s]$  such that

$$y = \frac{f(s) - f(a)}{s - a} = f'(x).$$

If  $y$  lies between  $f_b(a)$  and  $f_b(b)$ , then an analogous argument (exploiting the continuity of  $f_b$ ) shows that there exists  $s$  in  $[a, b)$  such that

$$y = \frac{f(b) - f(s)}{b - s} = f'(x).$$

This completes the proof.

This proof is more convincing as the author has claimed in his paper in the sense that it avoids the  $\epsilon - \delta$ -arguments. Author has used the Carathéodory's theorem for the existence of the auxiliary functions  $f_a$  and  $f_b$ . See T. M. Apostol [1], Bartle and Sherbert [2] for example. Use of this theorem for the existence of the functions  $f_a$  and  $f_b$  may still be rather too much for an undergraduate students beginning real analysis course. The proof I am going to give in the next section uses monotonicity property of a differentiable function, and the two standard and familiar theorems intermediate value theorem and the Rolle's theorem from the first year calculus course. This also uses no  $\epsilon - \delta$ -arguments.

## 2. Another Proof of Darboux's Theorem

PROOF. We assume that  $f'(a) < \lambda < f'(b)$ , and consider a function  $F: [a, b] \rightarrow \mathbb{R}$  defined by  $F(x) = f(x) - \lambda x$ . Then  $F'(x) = f'(x) - \lambda$ , and find that  $F'(a) = f'(a) - \lambda < 0$  and  $F'(b) = f'(b) - \lambda > 0$ . This means that the function  $F$  is not monotonic on  $[a, b]$  in the sense that there exist  $x, y, z \in [a, b]$  such that  $x < y < z$  satisfying exactly one and only one of the following conditions (a) or (b).

- (a)  $F(x) < F(y)$  and  $F(y) > F(z)$  We prove the theorem for this case.  
 (b)  $F(x) > F(y)$  and  $F(y) < F(z)$ . Proof in this case follows by a similar argument.

Suppose (a) holds. Then we encounter three cases:

- (i)  $F(x) < F(z)$   
 (ii)  $F(z) < F(x)$   
 (iii)  $F(z) = F(x)$ .

Case(i): Assume that  $F(x) < F(z)$ . Then  $F(x) < F(z) < F(y)$ .  $F$  is differentiable on  $[a, b]$  because so is  $f$  by hypothesis. That is  $F$  is continuous as well on  $[a, b]$ . So the Intermediate Value theorem applies to  $F$  on  $[x, y] \subseteq [a, b]$  and we obtain  $d \in (x, y) \subseteq (a, b)$  such that  $F(d) = F(z)$ . Note that

$$a \leq x < d < y < z \leq b.$$

Now we can apply the Rolles theorem to  $F$  on the closed interval  $[d, z]$  and obtain  $c \in (d, z) \subseteq (x, z) \subseteq (a, b)$  such that  $F'(c) = 0$  which leads to  $f'(c) = \lambda$  as desired.

Case (ii): Assume that  $F(z) < F(x)$ . Then  $F(z) < F(x) < F(y)$ . The theorem then follows by arguments similar to that of case (i).

Case (iii): Assume that  $F(z) = F(x)$ . Then we can use the Rolle's theorem for  $F$  directly on  $[x, z]$  and obtain a  $c \in (x, z) \subseteq (a, b)$  such that  $F'(c) = 0$  leading to  $f'(c) = \lambda$  as desired.

We arrive at the same conclusion if (b) holds by similar argument. This completes the proof of the theorem. □

## References

1. Apostol, T. M., *Mathematical Analysis*, 2nd ed., Narosa Publishing House, India, 2002.
2. Bartle, Robert G., Sherbert, Donald R., 3rd ed., *Introduction to Real Analysis*, John Wiley and Sons, Inc., 2000.
3. Boas, R., *A Primer of Real Functions*, 4th ed., Carus Mathematical Monographs, no. 13, Mathematical Association of America, Washington, DC, 1996.
4. Bressaud, David, 1st ed., *A Radical Approach to Real Analysis*, The Mathematical Association of America, 1994.
5. Darboux, G., *Memoire sur les fonctions discontinues*, Ann. Sci. Scuola Norm. Sup. 4(1875)57-122.
6. Hardy, G. H., 1st ed., *A Course of Pure Mathematics*, Rough Draft Printing, Cambridge, 2007.
7. Howie, John M., *Real Analysis*, 1st ed., Springer Undergraduate Mathematics Series, Springer, 2001.

8. Krantz, S., *Real Analysis and Foundations*, Cambridge University Press, Cambridge, 1992, reprint of the (1952) 19th ed.
9. Oslen, Lars, *A New Proof of darboux's Theorem*, The American Mathematical Society, Vol. 111, No. 8(Oct., 2004), pp. 713-715.
10. Ross, Kenneth A., 1st ed., *Elementary Analysis: The Theory of Calculus*, Springer-Verlag, New York, 1991
11. Rudin, W. *Principles Of Mathematical Analysis.*, 3rd ed., New York, International series in pure and applied mathematics, 1976. Print.
12. Sohrab, Houshang H., *Basic Real Analysis*, Brkhäuser, Boston, 2003.
13. Spivak, M *Calculus*, 4th ed., Library of Congress Catalogue, Houston, 2008.
14. Wade, William R., 4th ed., *An Introduction to Analysis*, Pearson, New Jersey, 2010.

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